Quantum Information Methods for Many-Body Physics

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Exercise Sheet 4: Magic of random systems **Due: 16 June, 10:00**

Clifford gates and stabilizer states are fundamental for quantum error correction and fault-tolerant computing. However, alone, these systems are efficiently simulatable – as we proved in the lecture notes. Magic, or non-stabilizerness, is required to reach beyond classically simulatable regimes. In this exercise sheet, we compute the magic of random states focusing on the Pauli spectrum and stabilizer entropy. These are the analog of anticoncentration metrics, based on the overlap on the computational basis, to the operator space, where the reference basis is Pauli operators.

The key object are the moments of the distribution $p_P = \frac{\langle\Psi|P|\Psi\rangle^2}{D}$ for $|\Psi\rangle = U|0\rangle$ random states generated by a Haar unitary distributed matrix $U \in \mathcal{U}(d^N)$. The probability of the overlaps p_P, formally defined by

$$\mathcal{P}(w) = \mathbb{E}_{P\in\mathcal{P}_N}\mathbb{E}_{U\in\mathcal{U}(d^N)}\left[\delta\left(w - \frac{\langle\Psi|P|\Psi\rangle^2}{D}\right)\right] ,$$

is a distribution of distribution and is given by the Pauli spectrum distribution. We will recover this from the knowledge of all the moments.

1 p_P is a probability distribution (2 P)

Show that for any state $|\Psi\rangle$ in the Hilbert space, p_P is a normalized distribution, that is $0 \leq p_P \leq 1$ and $\sum_{P\in\mathcal{P}_N} p_P = 1$.
Hint: In 2-replica space, the swap operator can be expressed as $R_{(12)} = \frac{1}{D}\sum_{P\in\mathcal{P}_N} P\otimes P$. This will allow for simplifications.

2 Expression of the moments (3 P)

Use the properties of the δ-function to find an expression for the moments

$$\zeta_k \equiv \int_{-\infty}^{\infty} dw\, w^k \mathcal{P}(w) , \tag{1}$$

in terms of overlaps. Express this quantity as a trace $\zeta_m = \mathrm{tr}(A\mathrm{M}_m(B))$: what are A and B? For the k-th moment, how many replicas do we need?

3 Second moment (3 P)

Compute ζ_2 explicitly, recalling the Weingarten fundamental expression for generic $k \geq 1$

$$\mathrm{M}_k(B) = \sum_{\sigma,\tau\in\mathrm{S}_k} \mathrm{Wg}_{\sigma,\tau}(d) T_\sigma \mathrm{tr}(T_\tau^\dagger B) . \tag{2}$$

Hint: recall that the Pauli strings have the following key properties: (i) $P^2 = I$ and (ii) $\mathrm{tr}(P) = \delta_{P,I}$ where I is the identity operator.

4 Third to fifth moment, and beyond (6 P)

Write the explicit value of ζ_k for $k = 3, 4, 5$. Can you guess the generic formula for larger replica moments? *Hint: Separate the contributions from $P = I$ and $P \neq I$. The following formula for counting* *permutations with a given cycle structure may be useful. Given a permutation* $\sigma \in \mathrm{S}_r$*, let* m_i *denote the number of cycles of length* i *in its cycle type* $\lambda(\sigma)$*. For example, the cycle type (2,2,1,1,1) corresponds to* $m_1 = 3$*,* $m_2 = 2$*. The number of permutations with this cycle structure is:*

$$\mathcal{N}_\lambda = \frac{r!}{\prod_{i=1}^{r}(i^{m_i} m_i!)} \,, \tag{3}$$

where $0! = 1 = 0^0$.

5 [Optional] Generating function and Pauli-spectrum (4 P)

From the knowledge of all the moments, you can compute the generating function

$$F(z) = \sum_{k=0}^{\infty} (-z)^k \frac{\zeta_k}{k!} \tag{4}$$

which is nothing but the Laplace transform of $P(w)$, namely $F(z) = \int dw P(w) e^{-zw}$. Use the inverse Laplace transform to obtain the $P(w)$.