Quantum Information meets Many-Body Physics

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Exercise Sheet 1 **Due: TBA**

1 Permutation fluency (3 P)

a) **(1 P)** For each of the following pairs, compute $\sigma\tau$ and $\tau\sigma$, write the answer in cycle notation, and decide whether the two permutations commute:

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}, \quad \tau_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}; \quad \sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 2 & 3 & 5 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 3 & 4 & 1 \end{pmatrix}.$$

Explain the second answer using the supports of the two permutations.

b) **(1 P)** Consider $\rho = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 2 & 8 & 5 & 7 & 1 & 4 & 6 \end{pmatrix}$. Find ρ^{-1} in two-line notation. Then find the disjoint-cycle decomposition, the cycle structure $\lambda(\rho)$, and the number of cycles $\#(\rho)$. Use the cycle decomposition as an independent check of your inverse.

c) **(1 P)** Decompose $\eta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 8 & 2 & 4 & 9 & 10 & 5 & 7 & 6 & 3 & 1 \end{pmatrix}$ into disjoint cycles, including fixed points. Determine $\lambda(\eta)$ and $\#(\eta)$.

2 Invariance of the cycle structure under conjugation (2 P)

Show that conjugation preserves the cycle structure: $\lambda(\pi\sigma\pi^{-1}) = \lambda(\sigma)$ for all $\sigma, \pi \in \mathrm{S}_k$.

3 Contractions of permutation operators (6 P)

Let $A_1, \dots, A_k \in \mathrm{L}(\mathcal{H})$ and $\pi \in \mathrm{S}_k$.

a) **(2 P)** Starting from the definition of the trace in a product basis, prove

$$\mathrm{tr}\big[(A_1 \otimes \cdots \otimes A_k) R_\pi\big] = \prod_{(a_1 \dots a_\ell) \in \mathcal{C}(\pi)} \mathrm{tr}(A_{a_1} A_{a_\ell} \cdots A_{a_2}), \tag{1}$$

where $\mathcal{C}(\pi)$ is the set of disjoint cycles of π, including fixed points. Explain why the order inside each trace is reversed by our convention for R_π.

b) **(1.5 P)** Set every $A_j = \mathbb{1}$. Find the cycle decomposition and compute $\mathrm{tr}(R_\pi)$ for

$$\pi_1 = (1\,4\,2)(3), \qquad \pi_2 = (1\,2)(3\,4\,5), \qquad \pi_3 = (1)(2\,6\,4\,3\,5),$$
$$\pi_4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 7 & 2 & 1 & 5 & 3 & 4 \end{pmatrix}, \qquad \pi_5 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 8 & 5 & 2 & 4 & 1 & 7 & 6 \end{pmatrix}.$$

c) **(2.5 P)** Let k be even and set $\mathcal{O}_k = (A \otimes B)^{\otimes k/2}$, with no assumption that A and B commute. Use Eq. (1) to compute $\mathrm{tr}(R_\pi \mathcal{O}_k)$ for

$$(1\,2) \in \mathrm{S}_2; \qquad (1\,2)(3\,5)(4\,6), \quad (1\,5\,3\,2)(4\,6), \quad (1\,3\,5)(4\,2\,6) \in \mathrm{S}_6;$$
$$(2\,8)(1\,6)(7\,3\,5)(4), \quad (2\,8\,1)(6\,7\,3\,4\,5) \in \mathrm{S}_8\,.$$

Give the order of the factors inside every nontrivial trace explicitly.

4 Gram and Weingarten matrices (3 P)

Let G be the Gram matrix and $W = G^{-1}$ ($d \geq k$) the Weingarten matrix (Sec. 2.3 of the lecture notes). Order S_2 as $\iota, (12)$ and S_3 as $\iota, (12), (13), (23), (123), (132)$.

a) (0.5+0.5+1 P) Compute G for $k = 1$, $k = 2$ and $k = 3$.

b) (1 P) Compute W for $k = 1$ and $k = 2$.

c) (+2 P, optional) Compute W for $k = 3$. *Hint:* Use symbolic software to help you invert G.