Quantum Information meets Many-Body Physics

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Exercise Sheet 2 **Due: TBA**

1 Symmetric subspace projector (3 P)

Using Weingarten calculus, show that for every normalized $|\psi\rangle \in \mathbb{C}^d$

$$\int_{\mathrm{U}(d)} \left(U|\psi\rangle\langle\psi|U^\dagger\right)^{\otimes k} \mathrm{d}U = \frac{P_{\mathrm{Sym},k,d}}{\binom{d+k-1}{k}}.$$

Hint: Follow Example 2.8.

2 Moments of Haar-random states and Porter–Thomas (6 P)

Let $|\Psi\rangle = U|0\rangle$ with $U \in \mathrm{U}(d)$ Haar random, and

$$P(w) = \mathbb{E}_x \, \mathbb{E}_U \left[\delta\left(w - |\langle x|\Psi\rangle|^2\right)\right],$$

where $\mathbb{E}_x$ is the uniform average over the d computational basis states $|x\rangle$.

a) **(3 P)** Use the properties of the δ-function to express the moments $I_k = \int \mathrm{d}w \, w^k P(w)$ in terms of the amplitudes $\langle x|\Psi\rangle$. Write $I_k = \mathrm{tr}\left(A \, \mathsf{M}_k(B)\right)$: who are A and B?

b) **(3 P)** Compute I_k for all $k \geq 1$ and, to leading order for large d, the generating function

$$F(z) = \sum_{k\geq 0} \frac{(-z)^k}{k!} I_k = \int \mathrm{d}w \, P(w) \, e^{-zw}.$$

Use the inverse Laplace transform to obtain $P(w)$.

3 A 1-design that is not a 2-design (4 P)

Let μ_{P} be the uniform distribution on the four one-qubit Pauli unitaries $\{\mathbb{1}, X, Y, Z\}$.

a) **(1.5 P)** Prove that μ_{P} is an exact unitary 1-design, i.e.

$$\frac{1}{4} \sum_{P\in\{\mathbb{1},X,Y,Z\}} PAP^\dagger = \frac{\mathrm{tr}(A)}{2} \, \mathbb{1}.$$

Hint: $\{\mathbb{1}, X, Y, Z\}$ is a basis of the 2×2 matrices.

b) **(1 P)** Apply the same random Pauli to both copies of $|0\rangle\langle 0|$ and compute $\frac{1}{4}\sum_P (P|0\rangle\langle 0|P^\dagger)^{\otimes 2}$.

c) **(1.5 P)** For a Haar-random one-qubit pure state, use

$$\int |\psi\rangle\langle\psi|^{\otimes 2} \, \mathrm{d}\psi = \frac{\mathbb{1} + R_{(12)}}{2(2+1)} = \frac{P_{\mathrm{Sym},2,2}}{3}$$

to show that the operator in (b) is not the Haar second moment: exhibit a normalized two-qubit vector on which the two operators have different expectation values. Conclude that μ_{P} is not a unitary 2-design.

4 Informationally complete measurements (5 P)

Consider measurements in m bases $(\varphi_{i,j})_{i\in[d]}$, $j \in [m]$, with $E_{i,j}$, Born probabilities $p(i,j|\rho)$ and frame operator S as in Sec. 3.1 of the lecture notes. State reconstruction is only possible if the Born probabilities differ for any two states $\rho \neq \rho'$; the measurement is then called *informationally complete* (IC).

a) **(2.5 P)** Show that the measurement is IC if and only if every $X \in \mathbb{C}^{d\times d}$ is a linear combination $X = \sum_{i,j} x_{i,j} E_{i,j}$, i.e. the $E_{i,j}$ span $\mathrm{L}(\mathcal{H})$. *Hint:* Consider the linear map $V(x) := \sum_{i,j} x_{i,j} E_{i,j}$ from $\mathbb{C}^{d\times m}$ to $\mathrm{L}(\mathcal{H})$ and its adjoint $V^\dagger$ (bra-ket notation might be useful).

b) **(2.5 P)** Show that the measurement is IC if and only if S is invertible. *Hint:* Show that $S = \frac{1}{m} V V^\dagger$.