Quantum Information meets Many-Body Physics

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Exercise Sheet 2 – Solutions **Due: TBA**

1 Symmetric subspace projector (3 P)

Using Weingarten calculus, show that for every normalized $|\psi\rangle \in \mathbb{C}^d$

$$\int_{\mathrm{U}(d)} \left(U|\psi\rangle\langle\psi|U^\dagger\right)^{\otimes k} \mathrm{d}U = \frac{P_{\mathrm{Sym},k,d}}{\binom{d+k-1}{k}}.$$

Hint: Follow Example 2.8.

Solution

The integral is $\mathsf{M}_k(|\psi\rangle\langle\psi|^{\otimes k})$. Relabelling $x_{\tau^{-1}(j)} \to x_j$ leaves $\psi_{x_1}\cdots\psi_{x_k}$ unchanged, so

$$R_\tau|\psi\rangle^{\otimes k} = \sum_x \psi_{x_1}\cdots\psi_{x_k}|x_{\tau^{-1}(1)},\dots,x_{\tau^{-1}(k)}\rangle = |\psi\rangle^{\otimes k}, \quad \mathrm{tr}\left(R_\sigma^\dagger|\psi\rangle\langle\psi|^{\otimes k}\right) = \langle\psi|^{\otimes k}R_{\sigma^{-1}}|\psi\rangle^{\otimes k} = 1.$$

By the Weingarten expansion and Lemma 2.3(c),

$$\mathsf{M}_k(|\psi\rangle\langle\psi|^{\otimes k}) = \sum_\pi \Big(\sum_\sigma W_{\pi,\sigma}\Big) R_\pi = \frac{(d-1)!}{(d+k-1)!}\sum_\pi R_\pi = \frac{k!\,(d-1)!}{(d+k-1)!}\,P_{\mathrm{Sym},k,d} = \boxed{\frac{P_{\mathrm{Sym},k,d}}{\binom{d+k-1}{k}}}.$$

2 Moments of Haar-random states and Porter–Thomas (6 P)

Let $|\Psi\rangle = U|0\rangle$ with $U \in \mathrm{U}(d)$ Haar random, and

$$P(w) = \mathbb{E}_x\,\mathbb{E}_U\left[\delta\left(w - |\langle x|\Psi\rangle|^2\right)\right],$$

where $\mathbb{E}_x$ is the uniform average over the d computational basis states $|x\rangle$.

a) **(3 P)** Use the properties of the δ-function to express the moments $I_k = \int \mathrm{d}w\, w^k P(w)$ in terms of the amplitudes $\langle x|\Psi\rangle$. Write $I_k = \mathrm{tr}\left(A\,\mathsf{M}_k(B)\right)$: who are A and B?

b) **(3 P)** Compute I_k for all $k \geq 1$ and, to leading order for large d, the generating function

$$F(z) = \sum_{k\geq 0}\frac{(-z)^k}{k!}\,I_k = \int \mathrm{d}w\, P(w)\, e^{-zw}.$$

Use the inverse Laplace transform to obtain $P(w)$.

Solution

(a) Since $\int \mathrm{d}w\, w^k\delta(w-a) = a^k$, $I_k = \mathbb{E}_x\mathbb{E}_U|\langle x\,|U|0\rangle|^{2k}$. As in Example 2.8, left invariance makes every x give the same term. With $\mathrm{tr}(Y)^k = \mathrm{tr}(Y^{\otimes k})$ for $Y = |0\rangle\langle 0|\,U|0\rangle\langle 0|U^\dagger$,

$$I_k = \mathbb{E}_U\,|\langle 0\,|U|\,0\rangle|^{2k} = \mathrm{tr}\left(|0\rangle\langle 0|^{\otimes k}\,\mathsf{M}_k(|0\rangle\langle 0|^{\otimes k})\right), \qquad \boxed{A = B = |0\rangle\langle 0|^{\otimes k}}.$$

(b) By Ex. 1 (for $k > d$, $c_\pi = \mathcal{G}_{k,d}^{-1}$ still solves $a = Gc$ by Lemma 2.3(b), and all solutions give the same $\sum_\pi c_\pi R_\pi$), $\mathsf{M}_k(B) = P_{\mathrm{Sym},k,d}/\binom{d+k-1}{k}$, and $P_{\mathrm{Sym},k,d}|0\rangle^{\otimes k} = |0\rangle^{\otimes k}$. Hence

$$I_k = \binom{d+k-1}{k}^{-1} = \frac{k!\,(d-1)!}{(d+k-1)!} = \frac{k!}{d(d+1)\cdots(d+k-1)} \simeq \frac{k!}{d^k} \quad (d \gg k),$$

i.e. $I_k = \mathrm{IPR}_k/d$ (Example 2.8). Then $F(z) \simeq \sum_{k\geq 0}(-z/d)^k = d/(d+z)$, and

$$P(w) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{d}z\, e^{zw}\, \frac{d}{d+z} = \boxed{d\, e^{-dw}\, \Theta(w)}$$

($c > -d$; for $w > 0$ close the contour to the left, residue at $z = -d$; for $w < 0$ close to the right, giving 0).

3 A 1-design that is not a 2-design (4 P)

Let μ_{P} be the uniform distribution on the four one-qubit Pauli unitaries $\{\mathbb{1}, X, Y, Z\}$.

a) (1.5 P) Prove that μ_{P} is an exact unitary 1-design, i.e.

$$\frac{1}{4} \sum_{P\in\{\mathbb{1},X,Y,Z\}} PAP^\dagger = \frac{\mathrm{tr}(A)}{2}\, \mathbb{1}.$$

Hint: $\{\mathbb{1}, X, Y, Z\}$ is a basis of the 2×2 matrices.

b) (1 P) Apply the same random Pauli to both copies of $|0\rangle\langle 0|$ and compute $\frac{1}{4}\sum_P (P|0\rangle\langle 0|P^\dagger)^{\otimes 2}$.

c) (1.5 P) For a Haar-random one-qubit pure state, use

$$\int |\psi\rangle\langle\psi|^{\otimes 2}\, \mathrm{d}\psi = \frac{\mathbb{1} + R_{(12)}}{2(2+1)} = \frac{P_{\mathrm{Sym},2,2}}{3}$$

to show that the operator in (b) is not the Haar second moment: exhibit a normalized two-qubit vector on which the two operators have different expectation values. Conclude that μ_{P} is not a unitary 2-design.

Solution

(a) Since $\mathrm{tr}(PQ) = 2\delta_{P,Q}$, $A = \frac{1}{2}\sum_Q \mathrm{tr}(QA)\, Q$. Two Paulis commute or anticommute, so $PQP^\dagger = \pm Q$, with $+$ iff $P \in \{\mathbb{1}, Q\}$ or $Q = \mathbb{1}$. Hence

$$\tfrac{1}{4}\textstyle\sum_P PQP^\dagger = \delta_{Q,\mathbb{1}}\, \mathbb{1} \quad (\text{for } Q \neq \mathbb{1}:\ \tfrac{1}{4}(1+1-1-1)Q = 0), \qquad \boxed{\tfrac{1}{4}\textstyle\sum_P PAP^\dagger = \tfrac{1}{2}\,\mathrm{tr}(A)\,\mathbb{1}}.$$

(b) $\mathbb{1}|0\rangle, Z|0\rangle \propto |0\rangle$ and $X|0\rangle, Y|0\rangle \propto |1\rangle$, so $\boxed{\frac{1}{4}\sum_P (P|0\rangle\langle 0|P^\dagger)^{\otimes 2} = \frac{1}{2}\big(|00\rangle\langle 00| + |11\rangle\langle 11|\big)}$.

(c) Take $|\Psi^+\rangle = (|01\rangle + |10\rangle)/\sqrt{2}$. Then $R_{(12)}|\Psi^+\rangle = |\Psi^+\rangle$ and

$$\tfrac{1}{2}\big(|\langle 00|\Psi^+\rangle|^2 + |\langle 11|\Psi^+\rangle|^2\big) = 0 \ \neq\ \tfrac{1}{6}\big(1 + \langle\Psi^+|R_{(12)}|\Psi^+\rangle\big) = \tfrac{1}{3}.$$

Since $\int |\psi\rangle\langle\psi|^{\otimes 2}\,\mathrm{d}\psi = \mathsf{M}_2(|0\rangle\langle 0|^{\otimes 2})$ ($|\psi\rangle = U|0\rangle$), the 2-design condition fails for $A = |0\rangle\langle 0|^{\otimes 2}$: μ_{P} is not a unitary 2-design.

4 Informationally complete measurements (5 P)

Consider measurements in m bases $(\varphi_{i,j})_{i\in[d]}$, $j \in [m]$, with $E_{i,j}$, Born probabilities $p(i,j|\rho)$ and frame operator S as in Sec. 3.1 of the lecture notes. State reconstruction is only possible if the Born probabilities differ for any two states $\rho \neq \rho'$; the measurement is then called *informationally complete* (IC).

a) **(2.5 P)** Show that the measurement is IC if and only if every $X \in \mathbb{C}^{d\times d}$ is a linear combination $X = \sum_{i,j} x_{i,j} E_{i,j}$, i.e. the $E_{i,j}$ span $\mathrm{L}(\mathcal{H})$. *Hint:* Consider the linear map $V(x) := \sum_{i,j} x_{i,j} E_{i,j}$ from $\mathbb{C}^{d\times m}$ to $\mathrm{L}(\mathcal{H})$ and its adjoint $V^\dagger$ (bra-ket notation might be useful).

b) **(2.5 P)** Show that the measurement is IC if and only if S is invertible. *Hint:* Show that $S = \frac{1}{m} VV^\dagger$.

Solution

(a) Let $e_{i,j}$ be the standard basis of $\mathbb{C}^{d\times m}$. Then $V = \sum_{i,j} |E_{i,j})e_{i,j}^\dagger$, so

$$V^\dagger = \sum_{i,j} e_{i,j}(E_{i,j}|, \qquad (V^\dagger X)_{i,j} = \mathrm{tr}(E_{i,j}X), \qquad p(\cdot|\rho) = V^\dagger \rho,$$

and $\ker V^\dagger = (\mathrm{ran}\, V)^\perp = \mathrm{span}\{E_{i,j}\}^\perp$.
Span ⇒ IC: $\ker V^\dagger = \{0\}$, so $V^\dagger(\rho - \rho') = 0$ implies $\rho = \rho'$.
No span ⇒ not IC: take $0 \neq X \in \ker V^\dagger$. Since $E_{i,j}^\dagger = E_{i,j}$, $\mathrm{tr}(E_{i,j}X^\dagger) = \overline{\mathrm{tr}(E_{i,j}X)} = 0$, so $X^\dagger \in \ker V^\dagger$. Hence one of $\frac{1}{2}(X + X^\dagger)$, $\frac{1}{2\mathrm{i}}(X - X^\dagger)$ is a Hermitian $H \neq 0$ in $\ker V^\dagger$, with $\mathrm{tr}\, H = \sum_i \mathrm{tr}(E_{i,1}H) = 0$. Then

$$\rho_\pm = \frac{\mathbb{1}}{d} \pm \varepsilon H \geq 0, \quad 0 < \varepsilon \leq \frac{1}{d\|H\|_\infty}, \qquad \rho_+ - \rho_- = 2\varepsilon H \neq 0,$$

are two states with $p(i,j|\rho_+) - p(i,j|\rho_-) = 2\varepsilon\, \mathrm{tr}(E_{i,j}H) = 0$.
(b) Using $e_{i,j}^\dagger e_{k,l} = \delta_{ik}\delta_{jl}$,

$$VV^\dagger = \sum_{i,j} |E_{i,j})(E_{i,j}| = mS, \qquad (X\,|S|\,X) = \frac{1}{m}\|V^\dagger X\|^2.$$

Hence $\ker S = \ker V^\dagger$, and by (a) $\boxed{S \text{ invertible} \iff \ker V^\dagger = \{0\} \iff \text{IC}}$.