Quantum Information meets Many-Body Physics

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Exercise Sheet 3 **Due: TBA**

1 Explicit tomography of one qubit (4 P)

Write an unknown qubit state as

$$\rho = \frac{1}{2}(\mathbb{1} + r_x X + r_y Y + r_z Z).$$

For each $a \in \{x, y, z\}$, let $p_{a,+}$ be the probability of obtaining the $+1$ outcome when measuring the corresponding Pauli operator σ_a.

a) **(1.5 P)** Derive $r_a = 2p_{a,+} - 1$ and hence reconstruct ρ from the three measurement probabilities.

b) **(1.5 P)** Diagonalize ρ and prove that the reconstructed matrix is a physical density matrix if and only if $r_x^2 + r_y^2 + r_z^2 \leq 1$.

c) **(1 P)** Empirical frequencies from finitely many shots can give a vector $\widehat{\boldsymbol{r}}$ outside the Bloch ball. Give a simple projected estimator that is always physical, and explain why the need for this projection does not contradict unbiasedness of the unprojected linear estimator.

2 Pauli spectrum and magic of Haar-random states (6 P)

Let $\mathcal{H} = (\mathbb{C}^2)^{\otimes n}$, $d = 2^n$, and $|\Psi\rangle = U|0\rangle$, $|0\rangle \equiv |0\rangle^{\otimes n}$, with $U \in \mathrm{U}(d)$ Haar random. For each of the d^2 Pauli strings P (Eq. (237) of the lecture notes) let $p_P := \langle\Psi|P|\Psi\rangle^2/d$, and

$$\mathcal{P}(w) = \mathbb{E}_P \, \mathbb{E}_U \big[\delta(w - p_P)\big],$$

with $\mathbb{E}_P$ the uniform average over the d^2 Pauli strings.

a) **(1.5 P)** Show that, for every $|\Psi\rangle$, $0 \leq p_P \leq 1/d$ and $\sum_P p_P = 1$. *Hint:* $\mathsf{F} = \frac{1}{d}\sum_P P \otimes P$, which follows from Eq. (244).

b) **(1.5 P)** Express the moments $\zeta_k = \int \mathrm{d}w \, w^k \mathcal{P}(w)$ as $\zeta_k = \mathrm{tr}\left(A \, \mathsf{M}_m(B)\right)$: who are A and B? How many replicas m are needed?

c) **(3 P)** Compute ζ_2. *Hint:* $P^2 = \mathbb{1}$ and $\mathrm{tr}\, P = 0$ for $P \neq \mathbb{1}$. The stabilizer Rényi entropy $\mathcal{M}_2 := -\log\left(d\sum_P p_P^2\right)$ quantifies magic. Compute $-\log \mathbb{E}_U\left[d\sum_P p_P^2\right]$ and compare with the value of $\mathcal{M}_2$ for $|0\rangle^{\otimes n}$.

d) **(+2 P, optional)** Compute ζ_k for all k. *Hint:*

$$\sum_{\substack{\pi \in \mathsf{S}_{2k} \\ \text{all cycles of even length}}} d^{\#(\pi)} = (2k-1)!! \, d(d+2)\cdots(d+2k-2).$$

3 Correcting symmetric readout noise (5 P)

In a local Pauli-shadow experiment, suppose every ideal eigenvalue $s_i \in \{+1, -1\}$ is independently flipped by the detector with probability $\eta < 1/2$. The recorded value is $\widetilde{s}_i = \xi_i s_i$, where $\Pr(\xi_i = -1) = \eta$.

a) **(2 P)** Show that $\mathbb{E}[\widetilde{s}_i \mid s_i] = (1-2\eta)s_i$ and derive the corrected one-qubit snapshot

$$\widehat{\rho}_i^{(\eta)} = \frac{1}{2}\Big(\mathbb{1} + \frac{3\widetilde{s}_i}{1-2\eta}P_i\Big),$$

where $P_i \in \{X, Y, Z\}$ is the randomly selected measurement axis. Verify its unbiasedness by averaging first over readout noise, then over outcomes and axes.

b) **(2 P)** Let Q be a Pauli string of weight w. Write its corrected one-shot estimator explicitly and prove

$$\mathbb{E}\big[(\widehat{o}_Q^{(\eta)})^2\big] = \Big(\frac{3}{(1-2\eta)^2}\Big)^w.$$

Give the corresponding variance and sample-complexity upper bounds.

c) **(1 P)** If the correction factor is omitted, compute the expectation of the noisy estimator. Interpret the exponential dependence on w and the divergence as $\eta \to 1/2$.