Quantum Information Methods for Many-Body Physics

Xhek Turkeshi, Markus Heinrich

Exercise Sheet 1 **Due: 17 April, 10:00**

1 Products of permutations (2 P)

For each pair of permutations, compute their products $\rho = \sigma \cdot \tau$ and $\pi = \tau \cdot \sigma$. Deduce which are commuting.

a) (0.5 P)

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix}$$

b) (0.5 P)

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix} \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

c) (0.5 P)

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 2 & 3 & 5 \end{pmatrix} \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 3 & 4 & 1 \end{pmatrix}$$

d) (0.5 P)

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 5 & 6 & 4 \end{pmatrix} \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 6 & 2 & 4 & 1 \end{pmatrix}$$

2 Inverse of Permutations (1.5 P)

Compute the inverse σ^{-1} of the following permutations

a) (0.5 P)

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 3 & 2 & 5 & 1 \end{pmatrix}$$

b) (0.5 P)

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 2 & 3 & 7 & 6 & 4 & 1 \end{pmatrix}$$

c) (0.5 P)

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 2 & 8 & 5 & 7 & 1 & 4 & 6 \end{pmatrix}$$

3 Cycle Decomposition of Permutations (2.5 P)

For each permutation, find the cycle decomposition. Deduce the cycle structure and the number of cycles.

a) (0.5 P)

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 1 & 6 \end{pmatrix}$$

b) (0.5 P)

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 4 & 1 & 2 & 3 & 6 & 7 \end{pmatrix}$$

c) (0.5 P)

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 4 & 5 & 2 & 1 & 6 & 3 & 8 \end{pmatrix}$$

d) (0.5 P)

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 5 & 6 & 4 & 1 & 2 & 9 & 3 & 7 \end{pmatrix}$$

e) (0.5 P)

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 8 & 2 & 4 & 9 & 10 & 5 & 7 & 6 & 3 & 1 \end{pmatrix}$$

4 Invariance of Cycle Structure under Conjugation (2 P)

Show that conjugation preserves the cycle structure of a permutation. That is, for any $\sigma, \pi \in S_k$, prove that

$$\lambda(\pi\sigma\pi^{-1}) = \lambda(\sigma).$$

5 Traces of Permutation Representation (5 P)

Consider the natural representation R_σ of permutations $\sigma \in S_k$ acting on $\mathcal{H}^{\otimes k}$, with $\mathcal{H}$ a Hilbert space of dimension d. Compute the trace

$$G(\sigma) \equiv \mathrm{tr}(R_\sigma) \tag{1}$$

of the following permutations using two different methods: 1) Using the definition of the trace, and 2) using graphical calculus.

a) (1 P)

$$\sigma = (1,4,2)(3)$$

b) (1 P)

$$\sigma = (1,2)(3,4,5)$$

c) (1 P)

$$\sigma = (1)(2,6,4,3,5)$$

d) (1 P)

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 7 & 2 & 1 & 5 & 3 & 4 \end{pmatrix}$$

e) (1 P)

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 8 & 5 & 2 & 4 & 1 & 7 & 6 \end{pmatrix}$$